Radiation impedance matrices for rectangular interfaces within rigid baffles: Calculation methodology and applications

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The coupling of sound fields through a finite-sized aperture in a plane rigid baffle where the region (half-space) on one side is unbounded can be described by an integral equation which constitutes a boundary condition for the field on the other side of the aperture. Such a boundary condition, when the pressure and the normal velocity are expanded in basis functions defined over the aperture, can be recast into a matrix form relating the coefficients of the basis functions in the expansions, the principal feature being a matrix of fourfold (double-area) integrals analogous to those encountered in studies of radiation from flexible pistons in rigid baffles. A substantial analytical reduction to sums of single nonsingular integrals is derived for the elements of this radiation impedance matrix when the aperture is rectangular and the basis functions are expressible as a sum of products of exponential functions of the Cartesian coordinates of the aperture plane, with the exponential coefficients being arbitrary complex numbers. The validity of the result is substantiated by its reduction to previously published results for less general cases. Its utility is demonstrated with the example of diffraction by a square hole in a screen. *© 2002 Acoustical Society of America.* [DOI: 10.1121/1.1430684]

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I. INTRODUCTION

Many acoustical systems of topical interest involve interfaces, with an interface typically dividing two regions with distinct characteristics, although sometimes[1] interfaces are conceptually created to separate regions where different mathematical descriptions are used. Mathematical formulations involving interfaces typically lead to equations involving integrals over all or a portion of the interface. Derivations of such integral equations date back to Helmholtz,[2] Rayleigh,[3] Kirchhoff,[4] Kellogg,[5] and Maue.[6] A review and fresh derivations can be found in a monograph article by Pierce.[7] Modern acoustical literature makes extensive use of such integral equations, as is exemplified in recent papers by Koo, Ih, and Lee,[8] by Giordano and Koopmann,[9] by Ginsberg and McDaniel,[10] and by Cunefare and De Rosa.[11]

The present paper is concerned with when the interface is a plane surface (Fig. 1), for which only a finite portion is not rigid. On one side of the interface is a semi-infinite half-space, and on the other side is an acoustical system that need not be explicitly specified. The nonrigid portion of the plane is the active interface (here referred to simply as the aperture), and the examples that are treated in this paper are for when this aperture has a rectangular shape. However, the technique developed here could be applied to apertures of more general shape.

The principal mathematical entity that emerges during the development of the analysis within the paper is a fourfold integral of the general form

$$J_{\alpha,\beta}(k)=\int_A\int_{A'}\Phi_\alpha(x,y)\Psi_\beta(x',y')\,\frac{e^{ikR}}{R}\,dx'\,dy'\,dx\,dy, \tag{1}$$

where $R=[(x-x')^2+(y-y')^2]^{1/2}$ is distance between points on the aperture. Both of the area integrations extend over the area of the aperture. The functions $\Phi_\alpha(x,y)$ and $\Psi_\beta(x,y)$ are the αth and βth members of each of two sets of functions—each set having N members, with the number N possibly being ∞. The two sets can possibly be the same set. The article explains how matrices, where the (α,β)th element is proportional to $J_{\alpha,\beta}$, can arise in the analysis of individual problems within a wide class of acoustical problems, and it also discusses how the requisite fourfold integral can be evaluated.

The explicit evaluation of integrals of such a generic type has been discussed in many previous papers. The present authors have studied, for example, the work presented in papers by Snyder and Tanaka,[12] by Takahagi, Nakai, and Yamai,[13] by Li and Gibeling,[14] and by Leppington, Broadbent, and Heron.[15] Various numerical and analytical tricks are known and discussed in this literature for simplifying the numerical work of the integration for special cases of the aperture shape and for special forms of the functions $\Phi_\alpha(x,y)$ and $\Psi_\beta(x,y)$; the principal achievement in the current paper is the reduction of the integration to a finite sum of one-dimensional integrals, where the integration is over a finite region, where the integrand is finite, and where the real and imaginary parts of the integrand have a finite number of maxima and minima. This is also for a special case; in particular, the aperture must be of rectangular shape, and the functions $\Phi_\alpha(x,y)$ and $\Psi_\beta(x,y)$ must each be of the form of a finite sum of terms, each term of the form $e^{px}e^{qy}$, where the exponent coefficients (p and q) are (in general) complex numbers and differ from term to term. Such restrictions nevertheless allow the possibility for the $\Phi_\alpha(x,y)$ and $\Psi_\beta(x,y)$ to be functions that one would naturally use in the solution of many acoustic problems. (The reason for the restriction involving exponentials is so that one can exploit the property

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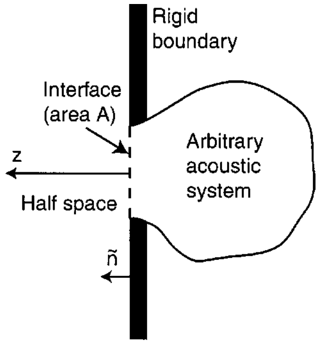

FIG. 1. Sketch of general situation for which the analysis of the present paper applies. An arbitrary acoustic system is coupled to a half-space through an interface of area *A*. The plane $[(x,y)$-plane$]$ of the interface except for the interface itself appears rigid to the medium on the other side $(z>0)$ of the half-space.

$e^u e^v = e^{u+v}$ in the analytic simplification of the integrals.)

Apart from the discussion of the methodology of the evaluation of such integrals, the significance of the present paper is that it shows how many complex problems can be reduced to a manageable form in which the presence of half-spaces is replaced by a boundary condition, with the boundary condition expressed by a matrix relation, in which integrals of the form of that in Eq. (1) appear in the matrix elements. Prior literature making use of such integrals has for the most part been restricted to the classic case of radiation from a baffled diaphragm in various specified states of vibration. The recognition that analogous mathematical ideas apply to wider classes of problems deserves a systematic exposition.

II. INTEGRAL RELATIONSHIPS AND MATRICES

A. Integral relations on interfaces

The region of the interface plane that is not rigid is the aperture of area *A*. On the other side of the area *A* there may be virtually anything insofar as the present paper is concerned. There may, for example, be an elastic plate with a fluid cavity on the other side. As viewed from the half-space, the interface is such that the normal component of the fluid velocity is zero along the rigid portion, but over the bounded area portion, it is in general nonzero. Insofar as the rest of the universe is concerned (i.e., that on the other side of the interface) the half-space can be formally replaced by an interface boundary condition. A suitable derivation results when the field in the half-space is written as the sum of an incident wave, a reflected wave, and a wave radiated from the aperture. The pressure associated with the reflected wave is taken as of the same form as for the incident wave, except (in accord with the method of images) the argument *z* is replaced by its negative. The combination of the incident and reflected waves conforms to the rigid surface boundary condition everywhere on the surface $z=0$. This boundary condition does not apply, however, on the aperture, where the outward normal component (away from the surface, into the fluid, and back toward the source) of the fluid velocity has some possibly nonzero value of $v_{n,\mathrm{int}}(x,y)$. This velocity, although not necessarily known, can be regarded as the source of a wave that radiates from the aperture back into the half-space. The expression for this wave results from an analysis due to Rayleigh,[3] the result of which is most frequently used in the prediction of sound radiation from baffled pistons. For the case described here, the taking of the limit as $z\rightarrow 0$ results in the following integral equation,

$$p_{\mathrm{int}}(\mathbf{x}_S) = 2p_{\mathrm{inc}}(\mathbf{x}_S) + \mathcal{M}(\mathbf{x}_S, \{v_{n,\mathrm{int}}(\mathbf{x}_S)\}), \tag{2}$$

on the portion *A* of the surface, where $\mathbf{x}_S$ is a point on the surface, p_{inc} is the amplitude of the incident wave, p_{int} is the pressure at the interface, and where

$$\mathcal{M}(\mathbf{x}_S, \{v_{n,\mathrm{int}}\}) = -\frac{i\omega\rho}{2\pi}\int\int v_{n,\mathrm{int}}(x',y')\frac{e^{ikR}}{R}\,dx'dy'. \tag{3}$$

Here *R* is the distance that appears in Eq. (1).

[This integral relation is a special case of Eq. (427) in the 1993 tutorial article by Pierce,[7] as the term $\mathcal{L}_1\{\zeta,p_{\mathrm{tot}}\}$ that appears there is identically zero when the bounding surface is flat. The general equation appears, possibly for the first time, as Eq. (10) in the 1949 paper by Maue.[6] For planar surfaces, the equation dates in principle back to an 1897 paper by Rayleigh,[16] insofar as Rayleigh used integral equations in relation to the problem of diffraction of sound by an aperture in a thin rigid screen. Although one can identify various instances where analogous ideas appear in the literature, the first explicit appearance of a version of Eq. (2) that is applicable to diffraction by an aperture is apparently Eq. (2.15) in the second edition (1950, the relevant passage being written by Copson) of Baker and Copson's monograph.[17] Equation (2) that appears above follows directly from Copson's Eq. (2.13) when one sets $x_o=0$.]

B. Integral relations as boundary conditions

A principal use for Eq. (2) is as a boundary condition for the portion of the overall acoustic system that lies on the other side of the aperture (i.e., that side that is not a half space). A nontrivial example is shown in Fig. 2. The mouth of the aperture is occupied by a cantilevered plate and this in turn is backed by a finite cavity with walls idealized as rigid. Because of the incident acoustic wave from the half-space side, both the backing cavity and the cantilevered plate are set into vibration. The partial differential equations of acoustics govern the fluid oscillations within the cavity, and the plate can be regarded as undergoing vibrations jointly forced by the pressure on the half-space side and by the pressure on the cavity side. The normal components of the fluid velocity on the two sides of the plate are equal and both are the same as the transverse velocity of the plate. The overall coupled vibration problem can be formulated with the integral relation (2) serving as the only requirement that relates the plate's transverse velocity to the pressure on its front side.

C. Matrix formulation of interface relationship

In the solution of specific problems, especially when the dimensions of the area *A* are less than a few wavelengths, the recasting of Eq. (2) into a matrix form provides a viable, although not necessarily exact, alternative. In the spirit of the

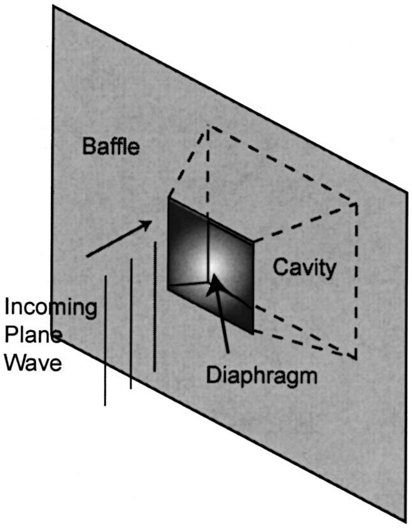

FIG. 2. Example of a problem to which the formulation in the paper applies. A plane wave is incident on a baffled cantilevered plate (displacements and slopes held to zero around the boundary), and the plate is backed by a cavity with rigid walls.

general Galerkin method, one chooses two sets of basis functions, $\{\Phi_\alpha(\mathbf{x}_S)\}$ and $\{\Psi_\beta(\mathbf{x}_S)\}$, each with N members, and assumes that sums of the form

$$p_{\rm int}=\sum_\alpha p_\alpha\Phi_\alpha ; \quad v_{n,\rm int}=\sum_\beta v_\beta\Psi_\beta \tag{4}$$

for appropriate choices of the coefficients, p_α and v_β, give adequate representations for the corresponding functions for all points within the area A. These individual basis functions for either set are not necessarily taken as orthogonal and normalized, but they are taken as linearly independent. (There are various rational ways for choosing these basis functions, some involving variational formulations and a priori physical insight. They can be taken as complete sets of orthonormal functions, with $N=\infty$, so that no approximation is necessarily implied. The implied arbitrariness here is to allow extensive latitude in the actual choice, although the choices are restricted in the subsequent discussion regarding the evaluation of integrals.)

With the expansions taken as valid, Eqs. (4) can be substituted into Eq. (2), and multiplication by any one of the $\Phi_\alpha(x,y)$, followed by integration over the area A, yields the set of N algebraic equations:

$$\sum_{\alpha'} N_{\alpha,\alpha'}p_{\alpha'}=2F_\alpha-\frac{i\omega\rho}{2\pi}\sum_\beta J_{\alpha,\beta}v_\beta , \tag{5}$$

where the quantity $J_{\alpha,\beta}$ is the integral that appears in Eq. (1). The other quantities that appear are

$$N_{\alpha,\alpha'}=\int_A \Phi_\alpha\Phi_{\alpha'}dA; \quad F_\alpha=\int_A p_{\rm inc}\Phi_\alpha dA. \tag{6}$$

Alternative recastings of the matrix relation (5) allow identification of what can be referred to as *radiation impedance* and *radiation admittance* matrices. In acoustical contexts the term impedance is used to refer to proportionality constants mapping velocity amplitudes to pressure amplitudes, while admittance is used to refer to proportionality constants mapping pressure amplitudes to velocity amplitudes. To identify the former, one multiplies both sides of (5) by the inverse of the (symmetric) matrix with elements $N_{\alpha,\alpha'}$, with the result

$$p_\alpha=2E_\alpha+\sum_\beta Z_{\alpha,\beta}v_\beta , \tag{7}$$

where E_α corresponds to the F_α term and

$$Z_{\alpha,\beta}=-\frac{i\omega\rho}{2\pi}\sum_{\alpha'}(N^{-1})_{\alpha,\alpha'}J_{\alpha',\beta} . \tag{8}$$

The latter is termed the *radiation impedance matrix* because

$$p_{\rm rad}(\mathbf{x}_S)=\sum_\alpha\left\{\sum_\beta Z_{\alpha,\beta}v_\beta\right\}\Phi_\alpha(\mathbf{x}_S) \tag{9}$$

is to be regarded (although possibly only as an approximation because of the truncation to finite N) as the radiated portion of the acoustic pressure at the aperture. (The linear independence of the basis functions Φ_α guarantees that the requisite matrix inverse exists.)

The *radiation admittance matrix*, with elements $Y_{\beta,\alpha}$, is analogously identified as the inverse of the radiation impedance matrix, so that

$$\sum_\alpha Y_{\beta,\alpha}Z_{\alpha,\beta'}=\delta_{\beta,\beta'} ; \quad \sum_\beta Z_{\alpha,\beta}Y_{\beta,\alpha'}=\delta_{\alpha,\alpha'} . \tag{10}$$

(The radiation impedance matrix is not necessarily symmetric; neither is the radiation admittance matrix. Nevertheless, right inverses are always the same as left inverses. The possibility that, with some choices of the two sets of basis functions, the radiation impedance matrix may not have an inverse is unlikely and is here disregarded.)

Pertinent results to be noted at this point are (*i*) that the boundary condition replacing the half-space can be expressed in terms of either the radiation impedance matrix or the radiation admittance matrix and (*ii*) evaluation of either of these matrices requires the evaluation of the integral in Eq. (1).

D. Fourier transform representation

In some of the applicable related literature, a recent example being a 1995 paper by Graham,[18] an alternate representation of the integral in Eq. (1) is used. Since the equivalence is not obvious and is typically not mentioned, a brief derivation is given here. A double Fourier transform of the kernel (Green's function evaluated at a point on the same plane as the source) that appears in the integral yields

$$\frac{e^{ikR}}{R}=-\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{N(\epsilon,k_x,k_y)}{(k^2-k_x^2-k_y^2)^{1/2}}dk_x\,dk_y , \tag{11}$$

with the numerator in the integrand being

$$N(\epsilon,k_x,k_y)=e^{-\epsilon k_x^2}e^{ik_x(x-x')}e^{-\epsilon k_y^2}e^{ik_y(y-y')}. \tag{12}$$

The radical in the denominator is understood to have a phase of $\pi/2$ whenever $k_x^2+k_y^2$ is greater than k^2.

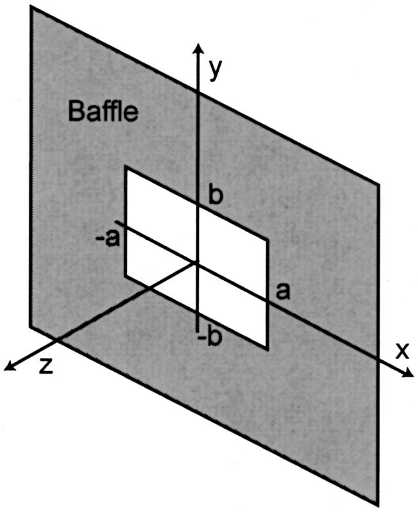

FIG. 3. Rectangular interface in a rigid baffle. The rectangle has dimensions $2a$ by $2b$, and the coordinate origin is in the center of the rectangle.

Insertion of the Fourier transform (11) into the quadruple integral (1), followed by an exchange of the order of integration, yields

$$J_{\alpha,\beta}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hat{\Phi}_{\alpha}(k_x,k_y)\hat{\Psi}_{\beta}(-k_x,-k_y)}{(k^2-k_x^2-k_y^2)^{1/2}}\,dk_x\,dk_y\,, \tag{13}$$

where

$$\hat{\Phi}_{\alpha}(k_x,k_y)=\int\int\Phi_{\alpha}(x,y)e^{ik_xx}e^{ik_yy}\,dA \tag{14}$$

with the overcaret designating the Fourier transform of the corresponding function. Here the implied limit as $\epsilon\rightarrow 0$ has been taken; the principal reason for expressing Eq. (11) as a limit is to guarantee that the interchange of integration order is allowable.

One could be tempted into taking Eq. (13) as a starting point rather than Eq. (1), inasmuch as that, for the same circumstances as are taken further below, the two double Fourier transforms $\hat{\Phi}_{\alpha}(k_x,k_y)$ and $\hat{\Psi}_{\beta}(-k_x,-k_y)$ can be evaluated in closed form, so the initial fourfold integral is immediately reduced to a twofold integral. However, the integrations over k_x and k_y are between infinite limits and one still has a singular integrand with which to contend. Nevertheless, this representation is useful when one considers high-frequency limits and can exercise the option of using asymptotic methods. The interest here is with when the frequency is not large.

III. A FUNDAMENTAL INTEGRAL FOR RECTANGULAR APERTURES

A. Generic basis functions

The circumstances alluded to above where some analytical simplification of the integral (1) is achievable are when the aperture is rectangular. For definiteness, the aperture is here taken (Fig. 3) to extend from $x=-a$ to $x=a$, and from $y=-b$ to $y=b$, so that the area A is $4ab$. (In much of the prior literature, a and b denote the total rectangle dimensions, and they accordingly correspond to quantities that are twice as large as the present paper's a and b.)

Among possible basis functions $\Phi_{\alpha}(x,y)$ or $\Psi_{\beta}(x,y)$ that one might use for such a geometry are products of Warburton[19] functions, one factor depending on x/a and the other factor depending on y/b, with the even and odd Warburton functions appropriate to the x-coordinate being

$$w_{e,n}(x/a)=\frac{\cosh(\kappa_{e,n}x/a)}{\cosh(\kappa_{e,n})}-\frac{\cos(\kappa_{e,n}x/a)}{\cos(\kappa_{e,n})}, \tag{15}$$

$$w_{o,n}(x/a)=\frac{\sinh(\kappa_{o,n}x/a)}{\sinh(\kappa_{o,n})}-\frac{\sin(\kappa_{o,n}x/a)}{\sin(\kappa_{o,n})}, \tag{16}$$

where $\kappa_{e,n}$ and $\kappa_{o,n}$ are roots of

$$\tanh(\kappa_{e,n})=-\tan(\kappa_{e,n});\quad \tanh(\kappa_{o,n})=\tan(\kappa_{o,n}). \tag{17}$$

These functions satisfy the boundary conditions of being zero and of having zero slope at $x=\pm a$ and jointly constitute a complete set. (They occur naturally in the theory of beams that are cantilevered at both ends.) Use of products of Warburton functions would be a natural choice if the aperture were occupied by a cantilevered plate. (The terminology ‘‘Warburton functions’’ is not standard, but there is no other name associated with the functions listed above. Warburton was apparently the first to recognize their potential usefulness as basis functions in more general contexts.)

Alternative choices of basis functions would be products of trigonometric functions, e.g., $\cos(n\pi x/a)$, $\cos([n+(\frac{1}{2})]\pi x/a)$, $\sin(n\pi x/a)$, and $\sin([n+(\frac{1}{2})]\pi x/a)$. One could choose a complete set of basis functions of such a type where the sets of x- and y-dependent factors are themselves complete sets, and where every element in, say, the set of x-dependent factors satisfies the boundary condition of being zero at the two end-points $(x=\pm a)$ or a complete set where every element satisfies the boundary condition of having zero derivative at the two end-points.

In all such considered cases, each basis function can be selected as a single product or as a sum of products of exponentials where a single term is a constant times a quantity of the generic form $e^{px/a}e^{qy/b}$. Here the quantities p and q, which differ from term to term, are constants, and can possibly be purely real, purely imaginary, or complex. Thus one has

$$\Phi_{\alpha}(x,y)=\sum_{\bar{n}}C_{\alpha,\bar{n}}e^{p_{\alpha,\bar{n}}x/a}e^{q_{\alpha,\bar{n}}y/b}, \tag{18}$$

$$\Psi_{\beta}(x,y)=\sum_{\bar{m}}D_{\beta,\bar{m}}e^{r_{\beta,\bar{m}}x/a}e^{s_{\alpha,\bar{m}}y/b}. \tag{19}$$

Here the sums should be regarded as sums over integers $\bar{n}$ and $\bar{m}$, respectively, with finite upper limits that depend on α and β, respectively. The quantities p and q depend on α and $\bar{n}$, while the quantities r and s depend on β and $\bar{m}$. The coefficients $C_{\alpha,\bar{n}}$ and $D_{\beta,\bar{m}}$ depend on the choice one has made for the basis functions, but are independent of position within the aperture. Note, for example, that any product and any sum of products of Warburton functions can be written in the form of Eqs. (18) and (19).

B. Definition of the fundamental integral

In the evaluation of the integrals that sum to yield the quantity in Eq. (1), a substantial notational simplicity results when one changes the integration variables to $\xi = x/a$ and $\eta = y/b$, thereby facilitating the definition $kR = \mathcal{R}$ with the identification

$$\mathcal{R}^2 = (ka)^2(\xi - \xi')^2 + (kb)^2(\eta - \eta')^2. \tag{20}$$

With such substitutions, the integral in (1) takes the form

$$J_{\alpha,\beta} = (ab)^{3/2} \sum_{\bar{n},\bar{m}} C_{\alpha,\bar{n}} D_{\beta,\bar{m}} I_4(p,q,r,s,ka,kb), \tag{21}$$

where

$$I_4(p,q,r,s,ka,kb) = k(ab)^{1/2} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{e^{i\mathcal{R}}}{\mathcal{R}} P \, d\xi' \, d\eta' \, d\xi \, d\eta, \tag{22}$$

$$P = e^{p\xi} e^{q\eta} e^{r\xi'} e^{s\eta'}. \tag{23}$$

In the latter definition, for notational brevity, the subscripts on *p, q, r, s* have been suppressed. The subscript 4 on I_4 serves as a reminder that the integral, as originally posed, is a fourfold integral. The coefficient in front of the integral has been selected so that I_4 is (*i*) dimensionless, (*ii*) symmetric in interchange of *p, r, ka* with *q, s, kb*, (*iii*) unchanged if *p* and *q* are replaced by *r* and *s*, respectively, and (*iv*) finite, neither zero nor infinite, in the limit $k \to 0$, with a/b held fixed. The achievement here is that the evaluation of the generic integral (1) has been reduced to the evaluation of an integral with the specific form of Eq. (22), the value of which is completely specified by six numbers.

The notation introduced above has the inconvenient property that it is not readily amenable to one's taking the limit of $k = 0$, with the constraint that a/b be held constant. However, the analysis given below leads to a natural decomposition

$$I_4(p,q,r,s,ka,kb) = J_1(ka,p,r,q,s,a/b) + J_1(kb,q,s,p,r,b/a). \tag{24}$$

Neither of the two terms on the right individually exhibits the symmetry property "(*ii*)" above, but the sum does have this property by virtue of the way the argument lists are written. [Identification of the function J_1 (which is a single, rather than a fourfold, integral) appears further below in Eq. (37) and in Eq. (42).]

C. Tutorial derivation of the analytical reduction

The initial step for evaluation of the integral I_4 is the transformation

$$\int_{-1}^{1}\int_{-1}^{1} Q(\xi,\xi') \, d\xi' \, d\xi = \int_0^2 F(u) \, du, \tag{25}$$

where

$$F(u) = \int_{-1+u}^{1} Q(\xi,\xi-u) \, d\xi + \int_{-1}^{1-u} Q(\xi,\xi+u) \, d\xi. \tag{26}$$

[This is derived by first breaking the ξ' integration into integrals from -1 to ξ and from ξ to 1. In the first such integral, one sets $\xi' = \xi - u$; in the second, one sets $\xi' = \xi + u$. In each case the integration variable becomes *u*, with the integration limits becoming 0 and $1+\xi$ for the first integral, and becoming 0 and $1-\xi$ for the second integral. In each of the resulting double integrals over ξ and *u*, the order of integration is changed so that the ξ integration is done first, the *u* integration is done second. In the first double integral, the integration is over a triangle with vertices ($u=0$, $\xi=1$), ($u=2$, $\xi=-1$), and ($u=0$, $\xi=1$), so the limits after the change of integration order become $-1+u$ and 1 for the ξ integration and 0 and 2 for the *u* integration. A similar interchange for the second double integral results in new integration limits of -1 and $1-u$ for the ξ integration and of 0 and 2 for the *u* integration.]

With an analogous transformation for the η and η' integrations, the integral I_4 becomes

$$I_4 = k(ab)^{1/2} \int_0^2 \int_0^2 \frac{e^{i\mathcal{R}}}{\mathcal{R}} A(u,p,r) A(v,q,s) \, du \, dv, \tag{27}$$

$$A(u,p,r) = B(u,p,r) + B(u,-p,-r) \tag{28}$$

$$B(u,p,r) = e^{-ru} \int_{-1+u}^{1} e^{[p+r]\xi} \, d\xi = \frac{1}{(p+r)} \left(e^{(p+r)} e^{-ru} - e^{-(p+r)} e^{pu} \right), \tag{29}$$

where now

$$\mathcal{R}^2 = (ka)^2 u^2 + (kb)^2 v^2. \tag{30}$$

[The result for the special case $p = -r$ follows directly (as discussed below) by setting $r = -p + \epsilon$ and then taking the limit as $\epsilon \to 0$, so it need not be considered separately. For typical choices of basis functions, this special case is likely.]

With the use of the splitting in Eq. (28), the integral I_4 breaks up into four terms, so one writes

$$\begin{aligned} I_4(p,q,r,s,ka,kb) &= \sum_{+,-} K_2(\pm(p,r), \pm(q,s), ka, kb) \\ &= K_2(p,r,q,s,ka,kb) + K_2(-p,-r,q,s,ka,kb) \\ &\quad + K_2(p,r,-q,-s,ka,kb) \\ &\quad + K_2(-p,-r,-q,-s,ka,kb), \end{aligned} \tag{31}$$

where

$$K_2(p,r,q,s,ka,kb) = k(ab)^{1/2} \int_0^2 \int_0^2 \frac{e^{i\mathcal{R}}}{\mathcal{R}} B(u,p,r) B(v,q,s) \, du \, dv. \tag{32}$$

Here the subscript 2 serves to remind one that the integral is a twofold integral. Note that the order of the exponent coefficients in the argument list of K_2 is different from that in the argument list of I_4. The reason is that the ensuing analysis tends to pair *p* with *r* and *q* with *s*. The summation convention implied by the notation in the first expression on the right side of Eq. (31) is defined by the second expression. One sums over all four possible sign combinations, but with p and r having their signs changed simultaneously, and with q and s having their signs changed simultaneously.

To reduce the double integral in Eq. (32) to a sum of single integrals, it is sufficient to transform the integration to polar coordinates. Some simplification results if one divides the square in the (u, v)-plane into two right triangles, each having a common hypotenuse along the line proceeding at an angle of 45° from the origin to the point (2,2). In the lower triangle, one sets

$$u=\frac{1}{ka}\mathcal{R}\cos\phi; \quad v=\frac{1}{kb}\mathcal{R}\sin\phi, \tag{33}$$

so that the domain of integration is

$$0<\mathcal{R}<2ka/\cos\phi; \quad 0<\phi<\tan^{-1}(b/a). \tag{34}$$

Analogous relations hold for the upper triangle, only with a and b interchanged. In both cases the differential of integration transforms to

$$du\,dv \to \frac{1}{k^2ab}\mathcal{R}d\mathcal{R}d\phi. \tag{35}$$

One also recognizes that the two integrals have identical form, providing one interchanges p and r with q and s, and ka with kb. Thus one can set

$$K_2(p,r,q,s,ka,kb)=L_1(ka,p,r,q,s,a/b)+L_1(kb,q,s,p,r,b/a). \tag{36}$$

Here arguments ka and a/b, or kb and b/a, are used because they appear more naturally in the derived expressions for the L_1 and because the L_1 are not symmetric in the interchange of p, r, ka with q, s, kb. Use of such arguments also makes it easy to take a meaningful limit as $k\to 0$. This allows identification of the function J_1 that appears in Eq. (24), the identification being

$$J_1(ka,p,r,q,s,a/b)=\sum_{+,-} L_1(ka,\pm(p,r),\pm(q,s),a/b), \tag{37}$$

where the convention for doing the sum is the same as in Eq. (31).

Performance of the $\mathcal{R}$-integration involved in the evaluation of the L_1 function yields

$$L_1(ka,p,r,q,s,a/b)=2(a/b)^{1/2}\int_0^{\tan^{-1}(b/a)}\Lambda\sec\phi\,d\phi. \tag{38}$$

Here the factor Λ is recognized (but see comments further below) as

$$\begin{aligned}\Lambda(ka,p,r,q,s,a/b,\phi) &= \frac{1}{(p+r)(q+s)}[e^{(p+r)}e^{(q+s)}F(ka,-r,-(a/b)s,\phi)\\ &\quad -e^{(p+r)}e^{-(q+s)}F(ka,-r,(a/b)q,\phi)\\ &\quad -e^{-(p+r)}e^{(q+s)}F(ka,p,-(a/b)s,\phi)\\ &\quad +e^{-(p+r)}e^{-(q+s)}F(ka,p,(a/b)q,\phi)]\end{aligned} \tag{39}$$

with

$$F(ka,p,(a/b)q,\phi)=\frac{e^{2D}-1}{2D}, \tag{40}$$

$$D(ka,p,(a/b)q,\phi)=ika\sec\phi+p+(a/b)q\tan\phi. \tag{41}$$

There is no convenient notational mnemonic for representing the sum of the four terms in Eq. (39), although a pattern is evident: the second term differs from the first in that $q\to -s$ and $s\to -q$; the third term differs from the first in that $r\to -p$ and $p\to -r$; and the fourth term differs from the first in that all of these sign changes and interchanges simultaneously take place.

Note that, with the result (38), the expression for the contributory term J_1 in Eq. (24) becomes

$$J_1(ka,p,r,q,s,a/b)=2(a/b)^{1/2}\int_0^{\tan^{-1}(b/a)}\left(\sum\Lambda\right)\sec\phi\,d\phi, \tag{42}$$

where all the similar terms with the same integration limits have been expressed as a single integral over the sum of the integrands; in this particular instance, the integrand factor is

$$\left(\sum\Lambda\right)=\sum_{+,-}\Lambda(ka,\pm(p,r),\pm(q,s),a/b). \tag{43}$$

Here, again, the summation convention is the same as in Eq. (31).

Equation (38) is applicable regardless of the values of the arguments. However, the explicit form (39) for the integrand $\Lambda(ka,p,r,q,s,a/b)$ applies only if both $p+r$ and $q+s$ are nonzero. Special forms of $\Lambda(ka,p,r,q,s,a/b)$ that are valid when one or both of these quantities are zero are derived in the following section.

In summary, the integral I_4 is given by Eq. (24) with the quantities J_1 given by Eq. (37), or equivalently by Eq. (42), and by the associated definitions that define the integrand. In the statement given here, it is understood that the indicated arguments are all dummy arguments. Thus, for example, the result for $L_1(kb,q,s,-p,-r,b/a)$ is obtained with suitable substitutions for arguments in the stated expression for $L_1(ka,p,r,q,s,a/b)$.

Equations (24), (42), and (43) compose the principal result of the present paper. The achievement is that the original fourfold singular integral has been reduced to a sum of two nonsingular single integrals. Moreover, providing $p+r\neq 0$ and $q+s\neq 0$, each such integral is representable in turn as a weighted sum of integrals, each of the generic form

$$W(ka,p,q,a/b)=\int_0^{\tan^{-1}(b/a)}\frac{e^{2D}-1}{2D}\sec\phi\,d\phi, \tag{44}$$

$$D=ika\sec\phi+p+(a/b)q\tan\phi, \tag{45}$$

and each depending on four (rather than six) numerical constants. This can be rewritten in a variety of alternate ways, but a reduction of the overall integral W to an analytical expression does not appear to be possible. Nevertheless, its direct numerical computation should present no difficulties. (Note that, in spite of the factor D in the denominator, the integrand is finite over the range of integration, regardless of the values of the four parameters. Three other generic integrals are needed to cover the special cases mentioned above—these are defined further below.)

D. Limiting cases

The above results strictly apply only if $p \neq -r$ and $q \neq -s$, but the special cases when one or both of the conditions $p=-r$ or $q=-s$ apply can be handled by taking limits. Thus, should the second and third arguments in the expression (39) for Λ be equal and opposite, one sets $r=-p+\epsilon$ and then takes the limit as $\epsilon \rightarrow 0$. Doing so yields

$$\begin{aligned}
&\Lambda(ka,p,-p,q,s,a/b,\phi)\\
&\quad =\frac{1}{(q+s)}\Bigg[2e^{(q+s)}F(ka,p,-(a/b)s,\phi)\\
&\qquad +e^{(q+s)}\left\{\frac{\partial}{\partial r}F(ka,-r,-(a/b)s,\phi)\right\}_{r=-p}\\
&\qquad -2e^{-(q+s)}F(ka,p,(a/b)q,\phi)\\
&\qquad -e^{-(q+s)}\left\{\frac{\partial}{\partial r}F(ka,-r,(a/b)q,\phi)\right\}_{r=-p}\Bigg],
\end{aligned} \tag{46}$$

where it is presumed that the other pair of exponential coefficients is *not* equal and opposite. An analogous result, but one involving differentiation with respect to s, emerges should the fourth and fifth arguments be equal and opposite. If both pairs are equal and opposite, one takes an additional limit and a second derivative emerges in the result.

In all instances, the indicated first and second differentiations yield as factors the generic functions

$$G(ka,p,(a/b)q,\phi)=\frac{\partial}{\partial D}\left\{\frac{e^{2D}-1}{2D}\right\} =\frac{1}{2D^2}(2De^{2D}-e^{2D}+1), \tag{47}$$

$$H(ka,p,(a/b)q,\phi)=\frac{\partial^2}{\partial D^2}\left\{\frac{e^{2D}-1}{2D}\right\} =\frac{1}{D^3}(2D^2e^{2D}-2De^{2D}+e^{2D}-1), \tag{48}$$

where the function $D(ka,p,(a/b)q,\phi)$ is the same as is defined in Eq. (44). With this notation, Eq. (46) and its counterparts that result for other limiting cases are rewritten as

$$\begin{aligned}
&\Lambda(ka,p,-p,q,s,a/b,\phi)\\
&\quad =\frac{1}{(q+s)}[2e^{(q+s)}F(ka,p,-(a/b)s,\phi)\\
&\qquad -e^{(q+s)}G(ka,p,-(a/b)s,\phi)\\
&\qquad -2e^{-(q+s)}F(ka,p,(a/b)q,\phi)\\
&\qquad +e^{-(q+s)}G(ka,p,(a/b)q,\phi)],
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\Lambda(ka,p,r,q,-q,a/b,\phi)\\
&\quad =\frac{1}{(p+r)}[2e^{(p+r)}F(ka,-r,(a/b)q,\phi)\\
&\qquad -(a/b)(\tan\phi)e^{(p+r)}G(ka,-r,(a/b)q,\phi)\\
&\qquad -2e^{-(p+r)}F(ka,p,(a/b)q,\phi)\\
&\qquad +(a/b)(\tan\phi)e^{-(p+r)}G(ka,p,(a/b)q,\phi)],
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\Lambda(ka,p,-p,q,-q,(a/b)q,\phi)\\
&\quad =4F(ka,p,(a/b)q,\phi)-2G(ka,p,(a/b)q,\phi)\\
&\qquad -2(a/b)(\tan\phi)[2G(ka,p,(a/b)q,\phi)\\
&\qquad -H(ka,p,(a/b)q,\phi)].
\end{aligned} \tag{51}$$

With these latter three equations, one has the mathematical apparatus for the full use of Eqs. (42) and (43), regardless of the specific values for the exponent factors p, q, r, and s. One may note that the totality of such contingencies is expressible in terms of four generic four-parameter integrals. One of these is the $W(ka,p,q,a/b)$ that appears in Eq. (44). The other three are

$$X(ka,p,q,a/b)=\int_0^{\tan^{-1}(b/a)}[2F-G]\sec\phi\, d\phi \tag{52}$$

$$Y(ka,p,q,a/b)=\int_0^{\tan^{-1}(b/a)}G\tan\phi\sec\phi\, d\phi, \tag{53}$$

$$Z(ka,p,q,a/b)=\int_0^{\tan^{-1}(b/a)}[2G-H]\tan\phi\sec\phi\, d\phi. \tag{54}$$

Here, for brevity, the argument lists of the integrand factors $2F-G$, G, and H have been suppressed. In all cases, the list is the same as appears in the definition of the quantities F and D in Eqs. (40) and (41), as well as in Eqs. (47) and (48). As is emphasized above, the arguments in the list of these defined functions should be regarded as dummy arguments.

IV. REDUCTION TO SIMPLER CASE OF RADIATION FROM A RIGID RECTANGULAR PISTON

A. Expression for the mechanical impedance

The principal intent here is to establish credibility, for readers not having the motivation to trace through in detail the totality of the mathematical steps above, that the derived results are indeed correct. To this purpose the case is examined of the total radiation force (area integral of pressure) on a rigid piston vibrating in a rigid baffle. This is perhaps the simplest instance in which Eq. (1) is encountered, and it has been often discussed in the literature. The interest here is specifically with results reported by Swenson and Johnson,[20] Chetaev,[21] Stenzel,[22] Nomura and Aida,[23] Burnett and Soroka,[24,25] Stepanishen,[26] and Levine.[27]

The ratio of force to velocity is ordinarily termed a *mechanical impedance*, $Z_{\mathrm{mech,rad}}$. In terms of the quantities derived in the previous section, and with the associated results incorporated, this mechanical impedance can be expressed so that

$$i\pi\frac{Z_{\text{mech,rad}}}{\rho cA}=\frac{k}{2A}\int\int\int\int\frac{e^{ikR}}{R}\,dx'\,dy'\,dx\,dy$$
$$=ka[2X(ka,0,0,a/b)-(a/b)Z(ka,0,0,a/b)]$$
$$+kb[2X(kb,0,0,b/a)-(b/a)Z(kb,0,0,b/a)].\tag{55}$$

Here the integrals $X(ka,0,0,a/b)$ and $Z(ka,0,0,a/b)$ are as defined above by Eqs. (40), (41), (47), (48), (52), and (54). In this limiting case, however, when the second and third arguments are zero, the quantity D is simply $ika\sec\phi$. For notational convenience, these integrals are here reexpressed as

$$X(ka,0,0,a/b)=\int_0^{\tan^{-1}(b/a)}T_1(ika\sec\phi)\sec\phi\,d\phi\tag{56}$$

$$Z(ka,0,0,a/b)=\int_0^{\tan^{-1}(b/a)}T_2(ika\sec\phi)\tan\phi\sec\phi\,d\phi\tag{57}$$

with the identifications

$$T_1(D)=\frac{1}{2D^2}[e^{2D}-2D-1],\tag{58}$$

$$T_2(D)=\frac{1}{D^3}[De^{2D}-e^{2D}+D+1].\tag{59}$$

In what follows, these expressions are used to show that the results of the present article are consistent with what has been given previously in the literature.

B. Swenson and Johnson's formula

A letter to the editor by Swenson and Johnson,[20] published in 1952, gives a highly abbreviated derivation (with only a brief suggestion of the methodology) of results for square and rectangular pistons and quotes (without any derivation) a result, here denoted as $I_{\bar{a},\bar{b}}$, for the quadruple integral that appears here in Eq. (55). (The quantities $\bar{a}$ and $\bar{b}$ are equal to the present paper's $2a$ and $2b$.)

Since the Swenson and Johnson formula is in the form of an expansion in powers of k, the comparison begins first with the development of power series for the T_1 and T_2 that appear in Eqs. (56) and (57), these being identified as

$$T_1(ika\sec\phi)=-i\sum_{n=1}^{\infty}\frac{(2i)^n}{(n+1)!}(ka)^{n-1}\sec^{n-1}\phi,\tag{60}$$

$$T_2(ika\sec\phi)=-2i\sum_{n=1}^{\infty}\frac{(2i)^n n}{(n+2)!}(ka)^{n-1}\sec^{n-1}\phi.\tag{61}$$

Insertion of these into Eqs. (56) and (57), followed by term-by-term integration, yields, after some additional mathematical steps,

$$I_{\bar{a},\bar{b}}=4(4ab)^{3/2}\sum_{m=0}^{\infty}(-1)^m(4k^2ab)^m$$
$$\times\left[\frac{A_m}{(2m+2)!}+ik(4ab)^{1/2}\frac{B_m}{(2m+3)!}\right],\tag{62}$$

where

$$A_m=(a/b)^{m+(1/2)}\int_0^{\tan^{-1}(b/a)}\sec^{2m+1}\phi\,d\phi$$
$$+(b/a)^{m+(1/2)}\int_0^{\tan^{-1}(a/b)}\sec^{2m+1}\phi\,d\phi$$
$$-\frac{1}{2m+3}[([a/b]+[b/a])^{m+(3/2)}-(a/b)^{m+(3/2)}$$
$$-(b/a)^{m+(3/2)}],\tag{63}$$

$$B_m=(a/b)^{m+1}\int_0^{\tan^{-1}(b/a)}\sec^{2m+2}\phi\,d\phi$$
$$+(b/a)^{m+1}\int_0^{\tan^{-1}(a/b)}\sec^{2m+2}\phi\,d\phi$$
$$-\frac{1}{2m+4}[([a/b]+[b/a])^{m+2}$$
$$-(a/b)^{m+2}-(b/a)^{m+2}].\tag{64}$$

Apart from some minor cosmetic changes so as to make fuller use of dimensionless quantities, and the use of i instead of $-j$, the expressions above are the same as those given by Swenson and Johnson in their Eqs. (8)–(10). The precise correspondence is

$$\bar{A}_{2m}=(-1)^m(4ab)^{m+(3/2)}A_m,\tag{65}$$

$$\bar{B}_{2m}=(-1)^m(4ab)^{m+2}B_m,\tag{66}$$

in accord with Swenson and Johnson's result

$$I_{\bar{a},\bar{b}}=4\sum_{m=0}^{\infty}\frac{\bar{A}_{2m}k^{2m}}{(2m+2)!}+i4k\sum_{m=0}^{\infty}\frac{\bar{B}_{2m}k^{2m}}{(2m+3)!}\tag{67}$$

and with $\bar{a}=2a$, $\bar{b}=2b$. The overbars distinguish symbols used in their paper from those used in the present paper.

Swenson and Johnson's formulas were rewritten and used in numerical calculations in a 1971 paper by Sauter and Soroka.[28] As pointed out by Sauter and Soroka, all of the integrals over the powers of the secant can be evaluated in "closed" form. The values of the first four such integrals are relatively simple:

$$\int_0^{\tan^{-1}(b/a)}\sec\phi\,d\phi=\ln[(1+[b/a]^2)^{1/2}+b/a],\tag{68}$$

$$\int_0^{\tan^{-1}(b/a)}\sec^2\phi\,d\phi=b/a,\tag{69}$$

$$\int_0^{\tan^{-1}(b/a)} \sec^3\phi\, d\phi = \frac{b}{2a}[(1+[b/a]^2)^{1/2}+b/a] + \tfrac{1}{2}\ln[(1+[b/a]^2)^{1/2}+b/a], \tag{70}$$

$$\int_0^{\tan^{-1}(b/a)} \sec^4\phi\, d\phi = (b/a) + \tfrac{1}{3}(b/a)^3. \tag{71}$$

Expressions for integrals over higher-order powers of the secant can be derived by consistent use of the mathematical identities

$$(n-1)\sec^n\phi - (n-2)\sec^{n-2}\phi = \frac{d}{d\phi}(\tan\phi\sec^{n-2}\phi), \tag{72}$$

$$\frac{d}{d\phi}\tan\phi = \sec^2\phi; \quad \frac{d}{d\phi}\ln(\sec\phi + \tan\phi) = \sec\phi. \tag{73}$$

C. Entrained mass and low-frequency limit

Examination of the first two terms in the expansion in powers of the wave number k yields additional substantiation that the two derivations, that of the present paper and that of Swenson and Johnson,[20] in addition to agreeing with each other, are correct. One finds, in particular, that

$$A_0 = (a/b)^{1/2}\ln[(1+[b/a]^2)^{1/2}+b/a] + (b/a)^{1/2} \times \ln[(1+[a/b]^2)^{1/2}+a/b] - \tfrac{1}{3}[([a/b]+[b/a])^{3/2} - (a/b)^{3/2} - (b/a)^{3/2}], \tag{74}$$

$$B_0 = \tfrac{3}{2}, \tag{75}$$

so that in the limit of low frequencies, Eqs. (55) and (62) yield

$$Z_{\text{mech,rad}} \approx -i\omega M_{\text{ent}} + \frac{1}{2\pi}\rho c A^2 k^2, \tag{76}$$

where

$$M_{\text{ent}} = \frac{A_o}{\pi}\rho A^{3/2} \tag{77}$$

is identified as the *entrained mass* (the apparent mass whose inertia produces the reactive part of the mechanical impedance).

The second term in Eq. (76) is the low-frequency radiation resistance and this is manifestly correct because, if one begins with Eq. (55) and expands the integrand factor $R^{-1}e^{ikR}$ in a power series, the corresponding term in the power series expansion of the double area integral is kA^2, where A is the area of the rectangle. What is reassuring is that the same result emerges also after a rather intricate limiting process from the more general result derived in the present paper.

In regard to the entrained mass, one notes that, for a square aperture, where $a/b = b/a = 1$,

$$A_o = 2\ln(1+\sqrt{2}) - \tfrac{2}{3}[\sqrt{2}-1] = 1.4866. \tag{78}$$

This yields an entrained mass for the square piston which may be compared with that for the circular piston,

$$M_{\text{ent}} = 0.473\rho A^{3/2} \quad \text{(square)}, \tag{79}$$

$$M_{\text{ent}} = 0.479\rho A^{3/2} \quad \text{(circular)}, \tag{80}$$

where the numerical coefficient in the former is the numerical value of A_o/π and the latter coefficient is the numerical value of $8/(3\pi^{3/2})$. The close agreement of the two numerical coefficients is striking and in accord with Rayleigh's prediction[29,30] that the entrained mass for an elliptical aperture (orifice) is very nearly the same as that of a circular aperture with the same area. From the standpoint of the present paper, the agreement is a striking confirmation that the analytical steps described here are correct.

D. Integral expressions of Chetaev and Levine

Chetaev[21] was the first to reduce the multiple integral describing the mechanical radiation impedance of a rigid piston to a sum of single integrals, although his integrals had higher transcendental functions, $Si(z)$ and $Ci(z)$, within the integrands. In subsequent years, various authors succeeded in finding clearer derivations and in reexpressing his result so that the integrands did not involve higher transcendental functions and so that the integrals would be more amenable to numerical evaluation. The version selected here for comparison with the present paper's result is that which appears as Eq. (13) of a 1983 paper by Levine.[27]

To derive Levine's equation from the present article's Eqs. (55)–(59), change the integration variable from ϕ to ζ, where, for the integrals, $X(ka,0,0,a/b)$ and $Z(ka,0,0,a/b)$, that appear in Eqs. (56) and (57), one sets $\sec\phi = \zeta$, so that

$$X(ka,0,0,a/b) = -\frac{1}{2(ka)^2}\int_1^{(1+[b/a]^2)^{1/2}} e^{2ika\zeta}\frac{d\zeta}{\zeta^2(\zeta^2-1)^{1/2}} + \frac{i}{ka}\int_1^{(1+[b/a]^2)^{1/2}} \frac{d\zeta}{\zeta(\zeta^2-1)^{1/2}} + \frac{1}{2(ka)^2}\int_1^{(1+[b/a]^2)^{1/2}} \frac{d\zeta}{\zeta^2(\zeta^2-1)^{1/2}}, \tag{81}$$

$$Z(ka,0,0,a/b) = \frac{i}{(ka)^3}\int_1^{(1+[b/a]^2)^{1/2}} \frac{d\zeta}{\zeta^3} - \frac{1}{(ka)^2}\int_1^{(1+[b/a]^2)^{1/2}} \frac{d\zeta}{\zeta^2} - \frac{i}{(ka)^3}\int_1^{(1+[b/a]^2)^{1/2}} e^{2ika\zeta}\frac{d\zeta}{\zeta^3} - \frac{1}{(ka)^2}\int_1^{(1+[b/a]^2)^{1/2}} e^{2ika\zeta}\frac{d\zeta}{\zeta^2}. \tag{82}$$

With the exception of the first term in Eq. (81), all of the integrals that appear in Eqs. (81) and (82) can either be directly performed or else combined, via integrations by parts, into expressions that are integrable. Thus, one derives

$$kaX(ka,0,0,a/b)+kbX(kb,0,0,b/a)$$

$$=i\pi/2+\frac{1}{2kab}[a^2+b^2]^{1/2}$$

$$-\frac{1}{2ka}\int_1^{(1+[b/a]^2)^{1/2}}\frac{e^{2ika\zeta}}{\zeta^2(\zeta^2-1)^{1/2}}d\zeta$$

$$-\frac{1}{2kb}\int_1^{(1+[a/b]^2)^{1/2}}\frac{e^{2ikb\zeta}}{\zeta^2(\zeta^2-1)^{1/2}}d\zeta, \qquad (83)$$

$$(ka^2/b)Z(ka,0,0,a/b)+(kb^2/a)Z(kb,0,0,b/a)$$

$$=\frac{i}{2k^2ab}[e^{2ik[a^2+b^2]^{1/2}}-e^{2ika}-e^{2ikb}+1]$$

$$+\frac{1}{kba}[(a^2+b^2)^{1/2}-a-b]. \qquad (84)$$

With the results in Eqs. (83) and (84), the expression in Eq. (55) consequentially yields

$$\frac{Z_{\text{mech,rad}}}{\rho cA}=1-\frac{i}{\pi abk}(a+b)-\frac{1}{2\pi abk^2}$$

$$+\frac{1}{2\pi abk^2}(e^{2ika}+e^{2ikb}-e^{2ik(a^2+b^2)^{1/2}})$$

$$+\frac{i}{\pi ka}\int_1^{(1+[b/a]^2)^{1/2}}\frac{e^{2ika\zeta}}{\zeta^2(\zeta^2-1)^{1/2}}d\zeta$$

$$+\frac{i}{\pi kb}\int_1^{(1+[a/b]^2)^{1/2}}\frac{e^{2ikb\zeta}}{\zeta^2(\zeta^2-1)^{1/2}}d\zeta, \qquad (85)$$

which is identical to the result in Eq. (13) of the cited paper by Levine. His a and b are equal to the present paper's $2a$ and $2b$. [The presence of the factor $(\zeta^2-1)^{-1/2}$ in the integrands in the two integrals may seemingly belie the assertion that the present paper achieves a reduction to nonsingular integrals. However, the singularity above is illusory as it can be removed by rewriting this singular factor as $\zeta^{-1}(d/d\zeta)\times(\zeta^2-1)^{1/2}$ and then integrating by parts.]

V. DIFFRACTION BY A SQUARE APERTURE

A more stringent test of the algorithm described in the previous sections is provided by the example of the diffraction of sound by a square aperture (Fig. 4) in a rigid screen. This problem has been recently considered by Hongo and Serizawa[31] with the use of a relatively complicated procedure that is difficult to comprehend from the written paper. They do, however, present numerical results with which other results can be compared. The special case considered here is when the aperture is square and when the incident wave is at normal incidence. The restriction to normal incidence allows one to use only those basis functions that have the same symmetry as that of a square. Thus one can choose, for the basis functions that appear in Eq. (4), the following,

$$\Phi_\alpha(x,y)=\Psi_\alpha(x,y)$$

$$=\tfrac{1}{2}[\cos(n\pi x/a)\cos(m\pi y/a)$$

$$+\cos(n\pi y/a)\cos(m\pi x/a)], \qquad (86)$$

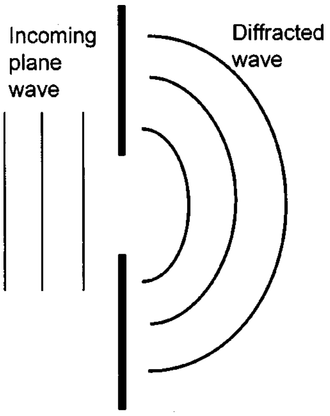

FIG. 4. Example of diffraction by a square aperture in a rigid screen. A plane wave at normal incidence impinges on a square orifice, and a diffracted wave emanates from the orifice on the other side of the screen.

where the integers $n(\alpha)$ and $m(\alpha)$ are to be regarded as functions of the integer α. An appropriate relationship that includes all possible basis functions of this type once and only once follows the pattern $n(1)=0$, $m(1)=0$; $n(2)=1$, $m(2)=0$; $n(3)=2$, $m(3)=0$; $n(4)=1$, $m(4)=1$; $n(5)=3$, $m(5)=0$; $n(6)=2$, $m(6)=1$; $n(7)=4$, $m(7)=0$; and $n(8)=3$, $m(8)=1$. The pattern of association is similar to the proof that one finds in some mathematics texts, as exemplified by the book by Courant and Robbins,[32] that the set of all pairs of integers is countable, or equivalently that the set of all rational numbers is countable, the original proof being due to Georg Cantor. Because the basis functions are symmetric in interchange of n and m, one can limit the set to just those when $n(\alpha)\geqslant m(\alpha)$, and progressively go down successive diagonals of a square of rows and columns, successive rows labeled by the values starting from 0 of the integer m, and successive columns labeled by the values starting from 0 of the integer n, so that $\alpha(0,0)=1$, $\alpha(1,0)=2$, $\alpha(2,0)=3$, $\alpha(1,1)=4$, $\alpha(3,0)=5$, etc., describes the inverse mapping from the pair (n,m) to $\alpha(n,m)$.

For the exponential expansions represented by either Eq. (18) or Eq. (19), the relevant constants, given the basis functions of Eq. (86), are identified such that $C_{\alpha,\overline{n}}=\frac{1}{8}$ for all $\overline{n}$, there being eight terms in the sum. The various pairs $(p_{\alpha,\overline{n}},q_{\alpha,\overline{n}})$ are the set (four in all) of the possible sign combinations of $(\pm in\pi/a,\pm im\pi/a)$ and the set (also four in all) of the possible sign combinations of $(\pm im\pi/a,\pm in\pi/a)$.

A convenient single number descriptor for the diffraction of the sound by the square aperture is the fraction of the incident power that is transmitted through the aperture. (Hongo and Serizawa refer to this quantity as the *transmission coefficient*.) If P_{inc} is the amplitude of the incident sound wave, then the incident time-averaged power is

$$[\text{Power}]_{\text{inc}}=\frac{1}{2}\frac{|P_{\text{inc}}|^2}{\rho c}A. \qquad (87)$$

For the plane screen diffraction problem, the pressure associated with the diffracted wave at the aperture is

$$p_{\text{diffr}}(\mathbf{x}_S)=-\mathcal{M}(\mathbf{x}_S,\{v_{n,\text{int}}(\mathbf{x}_S)\}) \qquad (88)$$

in accord with Rayleigh's result for the radiation from a vibrating portion of a plane and in accord with the definition that appears in Eq. (3). The minus sign on the right-hand side here is in accord with the previous definition of v_n as the component that points back toward the source. Since, along the interface, the continuity of pressure requires $p_{\text{diffr}} = p_{\text{int}}$, one concludes from Eqs. (2) and (88) that

$$p_{\text{diffr}}(\mathbf{x}_S) = p_{\text{inc}}(\mathbf{x}_S), \tag{89}$$

so the transmitted power is

$$\begin{aligned}[\text{Power}]_{\text{trans}} &= -\frac{1}{2}\text{Re}\int\int v_{n,\text{int}}(x,y)p^*_{\text{inc}}\,dx\,dy \\ &= -\frac{1}{2}\sum_{\beta}\ \text{Re}\left\{v_\beta\int\int p^*_{\text{inc}}\Psi_\beta\,dx\,dy\right\},\end{aligned} \tag{90}$$

where the quantities v_β are the solutions of the infinite set of equations

$$\sum_{\alpha'}\ (N^{-1})_{\alpha,\alpha'}\int\int p_{\text{inc}}\Phi_{\alpha'}\,dx\,dy = -\sum_{\beta}\ Z_{\alpha,\beta}v_\beta. \tag{91}$$

The selected basis functions are orthogonal and the incident pressure is uniform over the aperture, so the above relations reduce to

$$[\text{Power}]_{\text{trans}} = -\tfrac{1}{2}\text{Re}\{v_1P^*_{\text{inc}}A\}, \tag{92}$$

$$P_{\text{inc}}\delta_{\alpha,1} = -\sum_{\beta}\ Z_{\alpha,\beta}v_\beta. \tag{93}$$

The above relations, given the definition of the radiation admittance matrix in Eq. (10), reduce in turn to

$$[\text{Power}]_{\text{trans}} = \tfrac{1}{2}|P_{\text{inc}}|^2\text{Re}\{Y_{1,1}\}A, \tag{94}$$

where $Y_{1,1}$ is the corner element of the radiation admittance matrix. Thus, the fraction of the incident power that is transmitted is

$$\frac{[\text{Power}]_{\text{trans}}}{[\text{Power}]_{\text{inc}}} = \text{Re}\{Y_{1,1}\}\rho c. \tag{95}$$

The calculation of any element of the admittance matrix requires in principle that one invert a matrix (i.e., the radiation impedance matrix) with an infinite number of rows and columns. In practice, the computation of $Y_{1,1}$ is achieved by first defining $[Z_N]$ as the truncated impedance matrix, keeping only the square matrix formed from the first N rows and N columns. One defines $[Z_N]^{-1}$ as the inverse of this matrix. Then the appropriate identification of $Y_{1,1}$ is

$$Y_{1,1} = \lim_{N\to\infty}\ (\{[Z_N]^{-1}\}_{1,1}). \tag{96}$$

While one never takes such a limit with a computer, it can be inferred by simply plotting estimates resulting from successive values of N versus $1/N$ and extrapolating the plot to $1/N = 0$.

Figure 5 shows plots of the fraction of the incident power that is transmitted through the aperture versus the dimensionless frequency parameter *ka*. One plot has been transcribed from the $q = 1$ curve, q being b/a, of Fig. 6(a) (normal incidence) of the paper by Hongo and Serizawa;[31] the others result from the methodology of the current paper, as described above. The curves for $N = 10$ and $N = 66$ result from taking the approximate $Y_{1,1}$ as the (1, 1) element of the computed $[Z_N]^{-1}$ for the corresponding value of N. The fourth curve results from the extrapolation described by Eq. (96). The discrepancies are magnified somewhat by the choice of the vertical scale to a range fom 0.8 to 1.05. The most relevant observation from the standpoint of the present paper is that the curve based on the extrapolation agrees to within 0.5% with the curve taken from Hongo and Serizawa. At the time of this writing, it is not known which of the two curves is the more nearly correct, and there is no known third computation to adjudicate the discrepancy. What is here most important, however, is that the numerical results based on the paper's methodology depend on not just a single calculation of the integral in Eq. (1), but on each of N^2 elements, where N is up to 66, corresponding to $N^2 = 4056$. The fact that calculation of a continuous curve, each point depending on the calculation of so many fourfold integrals, is feasible testifies to the numerical speed-up achieved by the reduction of such integrals to a sum of integrations over a single variable. The good (even although not perfect) agreement with Hongo and Serizawa's results suggests furthermore that the mathematical analysis presented in earlier sections is indeed correct.

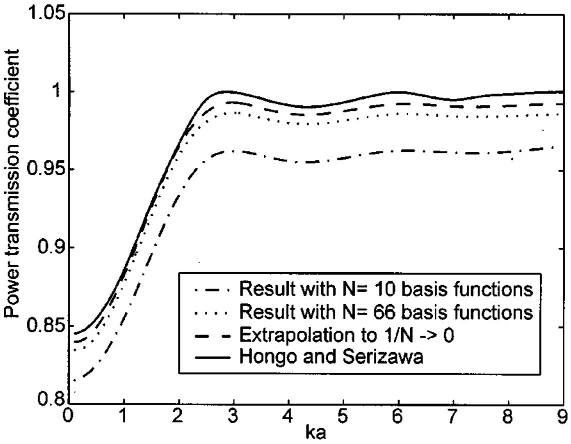

FIG. 5. Results of various approximate calculations for the power transmission coefficient versus frequency parameter *ka*. The case considered is that of a plane wave impinging at normal incidence on a square hole (dimensions $2a$ by $2a$) in a rigid screen. The solid curve is taken from a figure in a recent paper by Hongo and Serizawa (Ref. 31). The curves for $N = 10$ and $N = 66$ are based on estimates of the (1,1)-element of the radiation admittance matrix from the inverse of the truncated impedance matrix with N-rows and N-columns. The other curve results in the numerical extrapolation from finite N-approximations to when $1/N \to 0$. The power transmission coefficient is the fraction of the incident power on the hole that is transmitted to the diffracted wave on the other side of the screen.

[The seemingly slow convergence displayed by Fig. 5 of the curves for various N to the limiting case of $N \to \infty$ is a consequence of the somewhat simplistic choice of the basis functions Ψ_β represented by Eq. (86). It is known, from an analysis of the solution of Laplace's equation near a knife edge, that the normal velocity within the aperture must be singular at the edges, and that the singularity is an inverse square root singularity, so that, for example, $v_{\text{int},n} \sim 1/(a$

$-x)^{1/2}$ near $x=a$. The sum implied by the second of Eqs. (4) is accordingly not uniformly convergent, and rapid convergence of the computation represented by Eq. (96) is unlikely. An effort to speed-up this convergence by seeking a better set of basis functions that still conform to the constraint that each be expressible as a sum of products of exponentials, as in Eq. (19), was regarded as inappropriate for the present article. Other examples can be contemplated that illustrate the use of impedance matrix elements (with other choices of basis elements) for which either the matrix inversion is not necessary or for which more rapid convergence results, but the description of such examples will typically be longer.]

VI. CONCLUDING REMARKS

In practice, the principal results of this article, represented by the deduced equations in Sec. III, are relatively easy to program for numerical evaluation. The basic integrals, W, X, Y, Z, defined by Eqs. (44) and (52)–(54), are of the type that ordinarily presents negligible difficulties in numerical evaluation. One should be able to compute these integrals to any desired accuracy for zero to moderate values of ka. Numerical difficulties could arise for very large values of ka, $|p|$, or $|q|$, but in such cases asymptotic methods can be used. To limit the scope of the present paper, relevant asymptotic expressions are not given.

A possibly valid criticism of the overall methodology presented here is that the implied decompositions, as is exemplified, for example, by the sums in Eqs. (18) and (19), could lead to a moderately large (although certainly finite) number of terms. For specific cases, many of these terms can be analytically combined, and the number that one has to deal with could be drastically reduced. Here again, in the interest in restricting the scope of the article, examples dealing with such specific cases have been omitted. In practice, the computation of such a number of terms, even without analytical combinations, each only involving a single integration over one variable, will be quicker than evaluating a multiple integral, given a desired target of accuracy, especially if the target is not especially crude.

The applications that could make use of the formulation in Sec. II seem numerous, especially when one has in hand a convenient numerical method for calculating the individual elements of the radiation impedance matrix. We have previously studied the case of the radiation from a membrane at the mouth of a duct.[33] The listing and description of other applications that have occurred to the authors during the writing of this paper could be rather lengthy, but it is to be hoped that astute readers will perceive such applications and make use of the results of the present paper in some of their future research.

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